

Run-and-tumble particles, telegrapher's equation and absorption problems with partially reflecting boundaries

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Abstract. Absorption problems of run-and-tumble particles, described by the telegrapher's equation, are analyzed in one space dimension considering partially reflecting boundaries. Exact expressions for the probability distribution function in the Laplace domain and for the mean time to absorption are given, discussing some interesting limits (Brownian and wave limit, large volume limit) and different case studies (semi-infinite segment, equal and symmetric boundaries, totally/partially reflecting boundaries).

1. Introduction

Determining the first-passage time is a key issue in the analysis of many stochastic processes [1]. How long a signal takes to reach a threshold value or a diffusive particle to reach a target, to escape from a confined region through a small aperture, to be absorbed by a particular site of a boundary are very general questions which apply to different phenomena in the real world, from chemical reactions [2, 3] and intracellular transport [4] to animal movements [5] and financial time series [6]. Strictly speaking the problem is to determine the distribution of the time a particle takes to reach for the first time a given point. However, in many real situations the boundary is not perfectly absorbing and the particle can reach a given site many times before reacting with it. In such a case the relevant quantity is not the *first* time the particle reaches the site, but, instead, the time the particle *reacts* with it. Depending on the different contexts, such as, for example, absorption processes, chemical reactions, escaping from confined regions, one calls this quantity absorption time, reaction time, exit time, escape time. In the following we adopt the term *time to absorption* or *absorption time*, having in mind that, in the presence of partially absorbing boundaries, strictly speaking this is not the first time the particle reaches the site, but the first time the particle reacts with it (which are the same in the case of perfectly absorbing boundary). We are then dealing with *partially* reflecting (or absorbing) boundaries, the so called radiation boundary conditions. While such problems have been widely studied in the case of diffusive phenomena [1, 6, 7], a very good approximation for many physical and chemical processes, less attention, up to now, has been put on the non-diffusive case. This is the case, for example, of search processes with non-diffusive relocations in the presence of finite reaction rate [8], or processes involving active particles [9], like self-propelled cells [10], Janus particles [11–13] or molecular motors in microtubular filaments [14]. Active matter is becoming a relevant and growing field of research and allows us to extend many tools of statistical mechanics to the fascinating field of biological non-equilibrium processes [15]. Strictly related are the processes governed by the telegrapher's equation [16], which in 1D corresponds exactly to the run-and-tumble motion of a self-propelled particle. As an example, the firing of neurons can be modeled as a stochastic process with dichotomous noise, resulting in a telegrapher-like asymmetric equation [17]. First-passage time problems and solutions of the telegrapher's equation in the presence of totally absorbing and reflecting boundaries have been analyzed in the past [18–20]. More specifically, in Ref. [19] the solution for the probability density of the particle displacement is reported for one and two absorbing boundaries; different combinations of reflecting/absorbing boundaries are analyzed in Ref. [20]. Perfectly absorbing boundaries are also considered in a recent paper analyzing first-passage time problems of run-and-tumble particles [21]. Less attention, instead, has been paid to the problems with partially absorbing boundaries [16]. In Ref. [22] a solution of the telegrapher's equation is given, considering the special case of one partly absorbing point.

The aim of the present paper is to investigate absorption problems described by

the telegrapher's equation in the presence of partially reflecting boundaries. Exact expressions of the distribution (in the Laplace domain) of absorption time and its mean value will be given for the general situation of two different partially reflecting/absorbing boundaries, recovering some previously obtained results as particular cases.

The paper is organized as follow. In Sec. 2 the run-and-tumble model is introduced together with the appropriate boundary conditions. In Sec. 3 we define the time to absorption, giving a general expression for its mean value. In Sec. 4 and 5 two limiting cases are discussed, respectively the diffusive limit and the wave limit. In Sec. 6 different case studies are reported, giving exact expressions for the Laplace transformed probability distribution and the mean time to absorption. In Sec. 7 we discuss some features of the survival probability and absorption time distribution (higher moments, large volume limit) in the simplified case of symmetric boundaries. Conclusions are drawn in Sec. 8.

2. Run-and-tumble model and telegrapher's equation

We consider a run-and-tumble particle in one dimension confined in a finite interval $[a, b]$. The particle moves at constant speed v and performs tumble events at rate α , randomly reorienting its direction of motion. We consider here the case of instantaneous tumbling, a simplified description of realistic run-and-tumble bacteria [10]. The case of finite tumbling time can be treated in the framework of bimodal processes, and analytical results have been recently obtained for free run-and-tumble particles [23, 24]. Denoting with $P_R(x, t)$ and $P_L(x, t)$ the probability density function (PDF), respectively, of right-oriented and left-oriented particles, we can write the continuity equations as [21, 25–29]

$$\frac{\partial P_R}{\partial t} = -v \frac{\partial P_R}{\partial x} - \frac{\alpha}{2} P_R + \frac{\alpha}{2} P_L \quad (1)$$

$$\frac{\partial P_L}{\partial t} = v \frac{\partial P_L}{\partial x} + \frac{\alpha}{2} P_R - \frac{\alpha}{2} P_L \quad (2)$$

By introducing the total PDF P and the current J

$$P = P_R + P_L \quad (3)$$

$$J = v(P_R - P_L) \quad (4)$$

the equations for run-and-tumble particles can be written as

$$\frac{\partial P}{\partial t} = - \frac{\partial J}{\partial x} \quad (5)$$

$$\frac{\partial J}{\partial t} = -v^2 \frac{\partial P}{\partial x} - \alpha J \quad (6)$$

which correspond to the telegrapher's equation for P (or for J)

$$\frac{\partial^2 P}{\partial t^2} + \alpha \frac{\partial P}{\partial t} = v^2 \frac{\partial^2 P}{\partial x^2} \quad (7)$$

By using the Laplace transform in the time domain

$$\tilde{P}(x, s) = \int_0^\infty dt e^{-st} P(x, t) \quad (8)$$

the Eqs. (5,6) become

$$-P(x, 0) + s\tilde{P}(x, s) = -\frac{\partial \tilde{J}}{\partial x}(x, s) \quad (9)$$

$$-J(x, 0) + s\tilde{J}(x, s) = -v^2\frac{\partial \tilde{P}}{\partial x}(x, s) - \alpha\tilde{J}(x, s) \quad (10)$$

2.1. Initial conditions

We consider initial conditions

$$P_R(x, 0) = \frac{\delta(x)}{2} \quad (11)$$

$$P_L(x, 0) = \frac{\delta(x)}{2} \quad (12)$$

corresponding to equally distributed left and right oriented particles at the origin $x = 0$ (in the following we assume, without loss of generality, that $a < 0 < b$). Initial conditions for P and J read

$$P(x, 0) = \delta(x) \quad (13)$$

$$J(x, 0) = 0 \quad (14)$$

and Eq.s (9,10) become

$$\frac{\partial \tilde{J}}{\partial x} = \delta(x) - s\tilde{P} \quad (15)$$

$$v^2\frac{\partial \tilde{P}}{\partial x} = -(s + \alpha)\tilde{J} \quad (16)$$

which correspond to the second order differential equation for \tilde{P}

$$v^2\frac{\partial^2 \tilde{P}}{\partial x^2} - s(s + \alpha)\tilde{P} = -(s + \alpha)\delta(x) \quad (17)$$

In Sec. 3 we will extend the analysis to the case of general initial conditions.

2.2. Boundary conditions

We consider partially reflecting (partially absorbing) boundaries, with reflection coefficient γ (absorption coefficient $\eta = 1 - \gamma$). This corresponds to particles that, arriving at boundaries, can be either reflected, instantaneously reverting their direction of motion, or absorbed. It is worth noting that this does not encompass the situation in which particles can be stuck at a confining wall due to their persistent motion, which requires a different treatment. Boundary conditions at a and b can be written as [16,22]

$$P_R(a, t) = \gamma_a P_L(a, t) \quad (18)$$

$$P_L(b, t) = \gamma_b P_R(b, t) \quad (19)$$

where we have considered the possibility that the two boundaries can be different, with reflection coefficients γ_a and γ_b . In terms of P and J we have

$$J(a, t) = -\epsilon_a v P(a, t) \quad (20)$$

$$J(b, t) = \epsilon_b v P(b, t) \quad (21)$$

where we have introduced the coefficient ϵ

$$\epsilon = \frac{1 - \gamma}{1 + \gamma} = \frac{\eta}{2 - \eta} \quad (22)$$

For $\gamma = 0$ ($\eta = \epsilon = 1$) the boundary is perfectly absorbing, while for $\gamma = 1$ ($\eta = \epsilon = 0$) the boundary is perfectly reflecting.

In the Laplace domain boundary conditions read:

$$v \left. \frac{\partial \tilde{P}}{\partial x} \right|_{x=a} = (s + \alpha) \epsilon_a \tilde{P}(a, s) \quad (23)$$

$$v \left. \frac{\partial \tilde{P}}{\partial x} \right|_{x=b} = -(s + \alpha) \epsilon_b \tilde{P}(b, s) \quad (24)$$

The problem is then to find the solution of Eq. (17) with boundary conditions (23,24). The solution is of the form

$$\tilde{P}(x, s) = \begin{cases} A_1 e^{cx} + A_2 e^{-cx} & \text{for } x > 0 \\ A_3 e^{cx} + A_4 e^{-cx} & \text{for } x < 0 \end{cases} \quad (25)$$

where

$$v^2 c^2 = s(s + \alpha) \quad (26)$$

Coefficients A_i ($i = 1, \dots, 4$) are determined by imposing continuity of \tilde{P} and discontinuity of $\partial_x \tilde{P}$ – from Eq. (17) – at $x = 0$ and boundary conditions at $x = a$ and $x = b$, Eq.s (23,24).

3. Time to absorption

The survival probability, i.e. the probability that the particle has not yet been absorbed at time t , is

$$\mathbb{P}(t) = \int_a^b dx P(x, t) \quad (27)$$

The PDF of the absorption time is

$$\varphi(t) = -\frac{\partial \mathbb{P}}{\partial t}(t) \quad (28)$$

and its Laplace transform reads

$$\tilde{\varphi}(s) = 1 - s \int_a^b dx \tilde{P}(x, s) = 1 - s \tilde{\mathbb{P}}(s) \quad (29)$$

By using Eq. (15) it is easy to show that

$$\tilde{\varphi}(s) = \tilde{J}(b, s) - \tilde{J}(a, s) \quad (30)$$

or, in term of \tilde{P}

$$\tilde{\varphi}(s) = \epsilon_b v \tilde{P}(b, s) + \epsilon_a v \tilde{P}(a, s) \quad (31)$$

Table 1. Parameters of the model.

Quantity	Meaning
v	particle's speed
α	particle's tumbling rate
a, b	left and right position of boundaries ($a < 0 < b$)
ϵ_a, ϵ_b	absorption coefficients of boundaries [see Eq.(22)]

The time to absorption distribution is then obtained from the solution of Eq. (17) calculated at boundaries a and b . After some algebra one finally obtains

$$\tilde{\varphi}(s) = \frac{\epsilon_a(\epsilon_b + \frac{s}{vc})e^{cb} - \epsilon_a(\epsilon_b - \frac{s}{vc})e^{-cb} - \epsilon_b(\epsilon_a - \frac{s}{vc})e^{ca} + \epsilon_b(\epsilon_a + \frac{s}{vc})e^{-ca}}{(\epsilon_a + \frac{s}{vc})(\epsilon_b + \frac{s}{vc})e^{c(b-a)} - (\epsilon_a - \frac{s}{vc})(\epsilon_b - \frac{s}{vc})e^{-c(b-a)}} \quad (32)$$

or, rearranging the terms

$$\tilde{\varphi}(s) = \frac{\epsilon_a \epsilon_b v c (s + \alpha) (\sinh cb - \sinh ca) + s(s + \alpha) (\epsilon_b \cosh ca + \epsilon_a \cosh cb)}{[\epsilon_a \epsilon_b (s + \alpha) + s] v c \sinh c(b - a) + s(s + \alpha) (\epsilon_a + \epsilon_b) \cosh c(b - a)} \quad (33)$$

The meaning of the different parameters of the model are summarized in Table 1. It is worth noting that the quantity c is a function of s and depends on the parameters v and α through the Eq. (26).

The mean time to absorption τ is defined by

$$\tau = \int_0^\infty dt \, t \, \varphi(t) \quad (34)$$

and can be obtained from the derivative of $\tilde{\varphi}$

$$\tau = - \left. \frac{\partial \tilde{\varphi}}{\partial s} \right|_{s=0} \quad (35)$$

By using the previous expressions one finally obtains

$$\tau = \frac{1}{2v^2} \frac{(b-a)(2v^2 - \epsilon_a \epsilon_b a b \alpha^2) + \alpha v [\epsilon_a a^2 + \epsilon_b b^2 - 2ab(\epsilon_a + \epsilon_b)]}{v(\epsilon_a + \epsilon_b) + \epsilon_a \epsilon_b \alpha (b-a)} \quad (36)$$

The above expression has been obtained considering the initial condition $P(x, 0) = \delta(x)$, Eq. (13). It is easy to generalize it to the case of generic initial conditions. By considering particles starting at a generic point, $P(x, 0) = \delta(x - x_0)$, and explicitly writing the dependence on boundaries a, b and initial point $x_0 \in [a, b]$, we can write

$$\tau(a, b, x_0) = \tau(a - x_0, b - x_0, 0) \quad (37)$$

The average over generic initial conditions with distribution $p_{in}(x_0)$ is then obtained as

$$\langle \tau \rangle = \int_a^b dx_0 \, p_{in}(x_0) \, \tau(a - x_0, b - x_0, 0) \quad (38)$$

where the expression of τ inside the integral is that of Eq. (36) with the substitution $a \rightarrow a - x_0$ and $b \rightarrow b - x_0$. For example, considering uniform initial conditions $p_{in}(x_0) = 1/(b - a)$, one obtains

$$\langle \tau \rangle = \frac{b-a}{12v^2} \frac{12v^2 + \alpha(b-a)[4v(\epsilon_a + \epsilon_b) + \epsilon_a \epsilon_b \alpha (b-a)]}{v(\epsilon_a + \epsilon_b) + \epsilon_a \epsilon_b \alpha (b-a)} \quad (39)$$

4. Brownian limit

The Brownian (diffusion) limit is obtained by considering both large speed and tumble rate at constant diffusivity: $v \rightarrow \infty$, $\alpha \rightarrow \infty$, $D = v^2/\alpha$. The telegraph equation reduces to the diffusion (Fokker - Planck) equation

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2} \quad (40)$$

Partially reflecting boundaries for Brownian particles are described by the so called radiation boundary condition [1]

$$J(b, t) = kP(b, t) \quad (41)$$

where the coefficient k is in the range $[0, \infty)$, $k = 0$ describing the reflecting case and $k \rightarrow \infty$ the absorbing one. By comparing with Eq. (21) we then have that the correct limit is obtained considering a velocity dependent absorption coefficient η (or ϵ) which for large v behaves as $\eta \simeq 2k/v$ ($\epsilon \simeq k/v$) [30]. Possible choices are for example $\epsilon = \tanh(k/v)$ or $\epsilon = 1 - \exp(-k/v)$.

We then obtain the correct expression for the PDF $\tilde{\varphi}$ in the Brownian case from the general expression reported in the previous Section, Eq. (32), considering $v, \alpha \rightarrow \infty$ with $D = v^2/\alpha$ and $\epsilon v \rightarrow k$, $c^2 \rightarrow s/D$

$$\tilde{\varphi}_B(s) = \frac{k_a(k_b + \frac{s}{c})e^{cb} - k_a(k_b - \frac{s}{c})e^{-cb} - k_b(k_a - \frac{s}{c})e^{ca} + k_b(k_a + \frac{s}{c})e^{-ca}}{(k_a + \frac{s}{c})(k_b + \frac{s}{c})e^{c(b-a)} - (k_a - \frac{s}{c})(k_b - \frac{s}{c})e^{-c(b-a)}} \quad (42)$$

The mean time to absorption is obtained in the same limit from Eq. (36)

$$\tau_B = \frac{1}{2D} \frac{(b-a)(2D^2 - k_a k_b ab) + D[k_a a^2 + k_b b^2 - 2ab(k_a + k_b)]}{D(k_a + k_b) + k_a k_b(b-a)} \quad (43)$$

5. Wave limit

Another interesting limit is obtained for $\alpha \rightarrow 0$, i.e. considering the absence of tumbling. The telegraph equation in this limit reduces to the wave equation

$$\frac{\partial^2 P}{\partial t^2} = v^2 \frac{\partial^2 P}{\partial x^2} \quad (44)$$

In such a case a particle moves at constant velocity until it hits a barrier, where, with probability γ , reverses the direction of motion, and, with probability $1 - \gamma$, is absorbed. One obtains in such a case:

$$\tilde{\varphi}_W(s) = \frac{\epsilon_a(\epsilon_b + 1)e^{sb/v} - \epsilon_a(\epsilon_b - 1)e^{-sb/v} - \epsilon_b(\epsilon_a - 1)e^{sa/v} + \epsilon_b(\epsilon_a + 1)e^{-sa/v}}{(\epsilon_a + 1)(\epsilon_b + 1)e^{s(b-a)/v} - (\epsilon_a - 1)(\epsilon_b - 1)e^{-s(b-a)/v}} \quad (45)$$

The mean time to absorption takes the simple form

$$\tau_W = \frac{1}{v} \frac{b-a}{\epsilon_a + \epsilon_b} \quad (46)$$

The above expression can be also obtained by simple considerations. Indeed, in the absence of tumbling the particle bounces back and forth inside the box and each time

it reaches the boundary a (b) it is absorbed with probability η_a (η_b) and reflected with probability $\gamma_a = 1 - \eta_a$ ($\gamma_b = 1 - \eta_b$). If the particle starts its motion at the origin with right-oriented velocity, after a time $b/v + nL/v$ ($L = b - a$ and $n = 0, 1, \dots$) for even n it is absorbed by the boundary b with probability $\eta_b \gamma_b^{n/2} \gamma_a^{n/2}$, and for odd n it is absorbed by the boundary a with probability $\eta_a \gamma_b^{(n+1)/2} \gamma_a^{(n-1)/2}$. A particles starting with left-oriented velocity, at time $|a|/v + nL/v$ ($a < 0$) for even n it is absorbed by the boundary a with probability $\eta_a \gamma_a^{n/2} \gamma_b^{n/2}$, and for odd n it is absorbed by the boundary b with probability $\eta_b \gamma_a^{(n+1)/2} \gamma_b^{(n-1)/2}$. The mean time to absorption can then be obtained as an infinite sum

$$\tau_w = \frac{1}{2v} \sum_{n=0}^{\infty} [(b + nL) p_n(a, b) + (|a| + nL) p_n(b, a)] \quad (47)$$

where $p_n(a, b) = \eta_b \gamma_b^{n/2} \gamma_a^{n/2}$ for even n and $p_n(a, b) = \eta_a \gamma_b^{(n+1)/2} \gamma_a^{(n-1)/2}$ for odd n . The above sum gives exactly the result reported in Eq. (46).

6. Different boundaries: case studies

We analyze here some interesting cases, giving analytic expressions for the time to absorption distribution in the Laplace domain $\tilde{\varphi}(s)$ and the mean time to absorption τ .

6.1. Semi-infinite segment: $a \rightarrow -\infty$

In the case of a semi-infinite segment $(-\infty, b]$ the expression for the absorption time distribution can be obtained from Eq. (32) in the limit $a \rightarrow -\infty$

$$\tilde{\varphi}(s) = \frac{\epsilon_b}{\epsilon_b + \frac{s}{vc}} e^{-cb} \quad (48)$$

This situation has been treated in Ref. [22], where an exact expression for the Laplace transform of the density function $P(x, t)$ is reported and from which one can deduce the above expression for $\tilde{\varphi}(s)$. It is worth noting that the time to absorption diverges

$$\tau \rightarrow \infty \quad (49)$$

due to the open left boundary allowing particles to move far away from the absorbing wall. In the diffusion limit one has

$$\tilde{\varphi}_B(s) = \frac{k_b}{k_b + \frac{s}{c}} e^{-cb} \quad (50)$$

where $c^2 = s/D$. In the case of perfect absorption ($k \rightarrow \infty$) $\tilde{\varphi}_B(s) = \exp(-b\sqrt{s/D})$, giving rise to a survival probability – by performing inverse Laplace transform from Eq. (29) – $\mathbb{P}_B(t) = \text{erf}(b/\sqrt{4Dt})$, where $\text{erf}(x) = (2/\sqrt{\pi}) \int_0^x dy \exp(-y^2)$ [1, 31].

6.2. Equal boundaries: $\epsilon_a = \epsilon_b = \epsilon$

In the case of equal boundaries, with the same reflection coefficients, we have

$$\tilde{\varphi}(s) = \epsilon \frac{(\epsilon + \frac{s}{vc})e^{cb} - (\epsilon - \frac{s}{vc})e^{-cb} - (\epsilon - \frac{s}{vc})e^{ca} + (\epsilon + \frac{s}{vc})e^{-ca}}{(\epsilon + \frac{s}{vc})^2 e^{c(b-a)} - (\epsilon - \frac{s}{vc})^2 e^{-c(b-a)}} \quad (51)$$

and

$$\tau = \frac{b-a}{2v\epsilon} - \frac{\alpha ab}{2v^2} \quad (52)$$

In the diffusion limit the mean time to absorption reads

$$\tau_B = \frac{b-a}{2k} - \frac{ab}{2D} \quad (53)$$

6.3. Symmetric boundaries: $a = -b$

If the two boundaries are equidistant from the initial particle position (symmetric case), we have

$$\tilde{\varphi}(s) = \frac{[2\epsilon_a\epsilon_b + (\epsilon_a + \epsilon_b)\frac{s}{vc}]e^{cb} - [2\epsilon_a\epsilon_b - (\epsilon_a + \epsilon_b)\frac{s}{vc}]e^{-cb}}{(\epsilon_a + \frac{s}{vc})(\epsilon_b + \frac{s}{vc})e^{2cb} - (\epsilon_a - \frac{s}{vc})(\epsilon_b - \frac{s}{vc})e^{-2cb}} \quad (54)$$

and

$$\tau = \frac{1}{2v^2} \frac{2b(2v^2 + \epsilon_a\epsilon_b b^2 \alpha^2) + 3\alpha v b^2 (\epsilon_a + \epsilon_b)}{v(\epsilon_a + \epsilon_b) + 2\epsilon_a\epsilon_b \alpha b} \quad (55)$$

In the diffusion limit the mean time to absorption is

$$\tau_B = \frac{1}{2D} \frac{2b(2D^2 + k_a k_b b^2) + 3D b^2 (k_a + k_b)}{D(k_a + k_b) + 2k_a k_b b} \quad (56)$$

6.4. Equal and symmetric boundaries: $\epsilon_a = \epsilon_b = \epsilon$, $a = -b$

In the case of two equal and symmetric boundaries one has

$$\tilde{\varphi}(s) = \frac{2\epsilon}{(\epsilon + \frac{s}{vc})e^{cb} + (\epsilon - \frac{s}{vc})e^{-cb}} \quad (57)$$

and

$$\tau = \frac{b}{v\epsilon} + \frac{\alpha b^2}{2v^2} \quad (58)$$

A similar case, but considering totally absorbing boundaries ($\epsilon=1$), has been analyzed in previous papers [18, 20, 21]. The above expressions are particularly simple and will be taken as reference to discuss some other issues of absorption problems in the next Section. In the diffusion limit the mean time to absorption reads

$$\tau_B = \frac{b}{k} + \frac{b^2}{2D} \quad (59)$$

6.5. One reflecting boundary: $\epsilon_a = 0$

In the case of one perfectly reflecting boundary at $x = a$ and partially reflecting at $x = b$ one has

$$\tilde{\varphi}(s) = \epsilon_b \frac{e^{ca} + e^{-ca}}{(\epsilon_b + \frac{s}{vc})e^{c(b-a)} + (\epsilon_b - \frac{s}{vc})e^{-c(b-a)}} \quad (60)$$

and

$$\tau = \frac{b-a}{v\epsilon_b} + \frac{\alpha b}{2v^2}(b-2a) \quad (61)$$

It is worth noting that for $a = 0$ previous formulae reduce to those obtained in Sec. 6.4. Indeed, for symmetric reasons, the problem with two equal boundaries at $-b$ and b is equivalent to the one with a boundary at b and a reflecting boundary at the origin.

In the diffusion limit the mean time to absorption is

$$\tau_B = \frac{b-a}{k_b} + \frac{b}{2D}(b-2a) \quad (62)$$

Considering average over uniformly distributed initial conditions – see Eq. (39) – one has

$$\langle \tau_B \rangle = \frac{b-a}{k_b} + \frac{(b-a)^2}{3D} \quad (63)$$

in agreement with the expression reported in Table 1 of Ref. [32].

7. Discussion

In this Section I discuss some properties of the absorption time probability distribution, referring to the case of equal and symmetric boundaries, treated in the previous Section. This allows us to obtain simple analytical expressions without losing the underlying physics.

The expression for the (Laplace transformed) probability distribution of the absorption time in the case of boundaries with the same absorption properties ($\epsilon_a = \epsilon_b = \epsilon$) and located symmetrically with respect to the origin ($a = -b$), has been obtained in Section 6.4 and we rewrite it here for convenience

$$\tilde{\varphi}(s) = \frac{2\epsilon}{(\epsilon + \frac{s}{vc})e^{cb} + (\epsilon - \frac{s}{vc})e^{-cb}} = \frac{1}{\cosh cb + \frac{s}{\epsilon vc} \sinh cb} \quad (64)$$

In the following we discuss some features of this distribution.

7.1. Higher moments

All the moments (when they exist) of the distribution can be, in principle, obtained by simple derivatives

$$\langle t^n \rangle = (-1)^n \left. \frac{\partial^n \tilde{\varphi}}{\partial s^n} \right|_{s=0} \quad (65)$$

The first moment is the mean time to absorption, reported in the previous Section

$$\tau = \langle t \rangle = \frac{b}{v\epsilon} + \frac{\alpha b^2}{2v^2} \quad (66)$$

It is worth noting that the first term dominates when $b/\ell \ll 2/\epsilon$, where $\ell = v/\alpha$ is the particle's mean-free path. In other words, for small box length (with respect to ℓ) or also for moderately large box length in the case of weakly absorbing boundaries ($\epsilon \ll 1$), the mean time to absorption is dominated by the wave-like term. For large box length, instead, more precisely for $b/\ell \gg 2/\epsilon$, the second diffusive-like term dominates in Eq. (66). A more in-depth discussion of the large volume regime will be found at the end of this Section.

We report here also the expressions for the second moment

$$\langle t^2 \rangle = \frac{5}{3}\tau^2 + \frac{b^2}{v^2} \left(\frac{1}{3\epsilon^2} - 1 \right) \quad (67)$$

and the third moment

$$\langle t^3 \rangle = \frac{61}{15}\tau^3 + \frac{\alpha b^4}{2v^4} \left(\frac{9}{5\epsilon^2} - 5 \right) + \frac{b^3}{\epsilon v^3} \left(\frac{29}{15\epsilon^2} - 5 \right) \quad (68)$$

Generic expressions for higher moments will be given below in the large volume limit.

7.2. Survival probability

An interesting property of the survival probability $\mathbb{P}(t)$ can be inferred from the expression in Eq. (64), via the relation

$$\tilde{\mathbb{P}}(s) = \frac{1 - \tilde{\varphi}(s)}{s} \quad (69)$$

We first observe that, due to the finite speed of the particle, the quantity $\mathbb{P}(t)$ will be 1 up to the time b/v , i.e. the minimum time the particle takes to cover the distance b and then reach the boundary. We can then write $\mathbb{P}(t) = 1 - \theta(t - b/v) Q(t - b/v)$, having introduced the quantity $Q(t)$. Using the property

$$Q(0) = \lim_{s \rightarrow \infty} s \tilde{Q}(s) \quad (70)$$

we can obtain the value of the survival probability at time $t = b/v$

$$\mathbb{P}(t=b/v) = 1 - \frac{2\epsilon}{\epsilon + 1} \exp\left(-\frac{b\alpha}{2v}\right) \quad (71)$$

The survival probability $\mathbb{P}(t)$ has then a discontinuity at $t = b/v$, jumping from 1 to a lower finite value [19]. The previous expression has a simple physical interpretation: the quantity $\exp(-b\alpha/2v)$ is the probability that a particle has not changed its direction of motion up to time $t = b/v$ (thus ensuring that it reaches the boundary at time $t = b/v$) and $\eta = 2\epsilon/(\epsilon + 1)$ is the absorption probability. Hence the second term in the right-hand side of Eq. (71) is the probability that a particle is absorbed exactly at time $t = b/v$.

7.3. Large volume limit

We now analyze the behavior of the probability distribution φ in the large volume limit, $b \rightarrow \infty$. In this limit the mean time to absorption Eq. (66) diverges as

$$\tau \simeq \frac{\alpha b^2}{2v^2} \quad (72)$$

By defining the rescaled time $\hat{t} = t/\tau$, it is possible to show that, in the large volume limit, the probability distribution $\psi(\hat{t})$ has a universal simple form in the Laplace domain [33]

$$\tilde{\psi}(s) = \frac{1}{\cosh \sqrt{2s}} = \text{sech} \sqrt{2s} \quad (73)$$

By expanding in power of s we have

$$\tilde{\psi}(s) = \sum_{n=0}^{\infty} \frac{2^n E_{2n}}{(2n)!} s^n \quad (74)$$

where E_n are the Euler numbers. The generic moments of the distribution are then easily obtained from Eq. (65)

$$\langle \hat{t}^n \rangle = (-1)^n \frac{2^n n!}{(2n)!} E_{2n} \quad (75)$$

For example, the first three moments are 1, 5/3 and 61/15, which, due to the relation $\langle t^n \rangle = \tau^n \langle \hat{t}^n \rangle$, correspond to

$$\langle t \rangle = \frac{\alpha b^2}{2v^2} \quad (76)$$

$$\langle t^2 \rangle = \frac{5}{3} \tau^2 \quad (77)$$

$$\langle t^3 \rangle = \frac{61}{15} \tau^3 \quad (78)$$

in agreement with the large volume limit of Eq.s (66,67,68).

8. Conclusions

Absorption problems in the presence of partially reflecting boundaries have been analyzed for active particles animated by run-and-tumble dynamics in one space dimension (telegrapher's equation). General expressions are obtained for the absorption time distribution in the Laplace domain and for the mean time to absorption. Brownian and wave limits are retrieved, respectively, for $v, \alpha \rightarrow \infty$ and $\alpha \rightarrow 0$. Some interesting case studies are analyzed: the case of semi-infinite segment, the case of equal boundaries with the same reflection coefficients, the spatially symmetric case and the case of coexistence of totally and partially reflecting boundaries. Large volume limit is also discussed, giving explicit expressions for generic moments of the probability distribution of absorption time. Reported results can be applied to all the cases in which the system dynamics is described by a telegraph-like equation.

References

- [1] Redner S 2001 *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, UK)
- [2] Kramers H A 1940 *Physica* **7** 284
- [3] Hänggi P, Talkner P and Borkovec M 1990 *Rev. Mod. Phys.* **62** 251
- [4] Bressloff P S and Newby J M 2013 *Rev. Mod. Phys.* **85** 135
- [5] McKenzie H W, Lewis M A and Merrill E H 2009 *Bulletin of Mathematical Biology* **71** 107
- [6] Metzler R, Oshanin G and Redner S 2014 *First-Passage Phenomena and Their Applications* (World Scientific Publishing Company, Singapore)
- [7] Rupperecht J F, Bénichou O, Grebenkov D S and Voituriez R 2012 *J Stat Phys* **147** 891
- [8] Bénichou O, Loverdo O, Moreau M and Voituriez R 2006 *Phys. Rev. E* **74** 020102
- [9] Ramaswamy S 2010 *Annu. Rev. Condens. Matter Phys.* **1** 323
- [10] Berg H C 2004 *E. Coli in Motion* (Springer-Verlag, New York)
- [11] Golestanian R, Liverpool T B and Ajdari A 2005 *Phys. Rev. Lett.* **94** 220801
- [12] Walthera A and Müller A H E 2008 *Soft Matter* **4** 663
- [13] Jiang H-R, Yoshinaga N and Sano M 2010 *Phys. Rev. Lett.* **105** 268302
- [14] Sanchez T, Chen D T N, DeCamp S J, Heymann M and Dogic Z 2012 *Nature* **491** 431
- [15] Cates M E 2012 *Rep. Prog. Phys.* **75** 42601
- [16] Weiss G H 2002 *Physica A* **311** 381
- [17] Müller-Hansen F, Droste F and Lindner B 2015 *Phys. Rev. E* **91** 022718
- [18] Weiss G H 1984 *J. Stat. Phys.* **37** 325
- [19] Masoliver J, Porrà J M and Weiss G H 1992 *Phys. Rev. A* **45** 2222
- [20] Leydolt H.J. 1993 *Phys. Rev. E* **47** 3988
- [21] Angelani L, Di Leonardo R and Paoluzzi M 2014 *Eur. Phys. J. E* **37** 59
- [22] Masoliver J, Porrà J M and Weiss G H 1993 *Phys. Rev. E* **48** 939
- [23] Angelani L 2013 *EPL* **102** 20004
- [24] Detcheverry F 2014 *Eur. Phys. J. E* **37** 114
- [25] Schnitzer M J 1993 *Phys. Rev. E* **48** 2553
- [26] Tailleur J and Cates M E 2008 *Phys. Rev. Lett.* **100** 218103
- [27] Tailleur J and Cates M E 2009 *Europhys. Lett.* **86** 60002
- [28] Martens K *et al.* 2010 *Eur. Phys. J. E* **35** 84
- [29] Angelani L, Costanzo A and Di Leonardo R 2011 *EPL* **96** 68002
- [30] Erban R and Chapman S J 2007 *Phys. Biol.* **4** 16
- [31] Redner S and Krapivsky PL 1999 *Am. J. Phys.* **67** 1277
- [32] Szabo A, Schulten K and Schulten Z 1980 *J. Chem. Phys.* **72** 4350
- [33] Bénichou O and Voituriez R 2014 *Physics Reports* **539** 225